FINANSMINISTERIET

Reliability of output gaps estimates – comparing multivariate methods with traditional production function methods

2015

Working paper series no. 0X/2015 (preliminary draft)

September 2015

Reliability of output gaps estimates – comparing multivariate methods with traditional production function methods*

By

Anders Borup Christensen

[Preliminary draft]

Abstract

This paper describes the multi-variate approach used by the Danish Ministry of Finance (MoF) to estimate the output gap in Denmark and how the approach differs from the traditional production function approach used by for example the European Commission and the OECD. The paper uses synthetic data to analyze how different methods affect the precision and the real-time reliability of the estimates. Also, working with artificial data allows us analyze how the precision and reliability are affected by different degrees of data noise and misspecification of the estimation model. The results indicate that the particular features of the MoF-model improve the precision and the real-time reliability of the output gap estimates.

* The views expressed in this paper are those of the author(s) and do not necessarily represent the views of the Ministry of Finance.

1 Introduction

The output gap is a key variable of interest in policy making. But output gap estimates are surrounded by significant uncertainty and are subject to revision over time as more information becomes available. To be relevant for policy makers, the output gap estimate needs to be as accurate as possible while remaining stable over time, to ensure that the policy messages derived from the output gap calculations are stable over time to avoid that former policy advice will appear incorrect in hindsight. Thus, a desirable feature of output gap estimates is real-time reliability.

The Danish Ministry of Finance (MoF) uses a multi-variate state space model to estimate the output gap. The multi-variate approach allows one to combine Kalman Filter techniques with reduced form economic relationships and use survey data as an indicator for cyclical swings in the economy. Recent studies show (IMF Working Paper 15/144) that multi-variate estimation models outperform single-variate filtering techniques by reducing end-of-sample problems and thereby increasing real-time reliability.

In section 2 and 3 the MoF methodology for estimating the output gap is presented and compared to the more traditional production function method used by the European Commission (EC) and OECD.

Section 4 presents historical evidence that MoF output gap estimates for Denmark tend to be more stable over time than estimates by the EC and OECD. However, this evidence of past revisions of output gap estimates does not give us much clarity on which model properties that may be behind these features.

In section 5 we use of synthetic data to identify which key difference in the methodologies affect the precision and reliability of output gap estimates. The results in line with other studies suggest that some of the multi-variate features of the MoF methodology improve reliability.

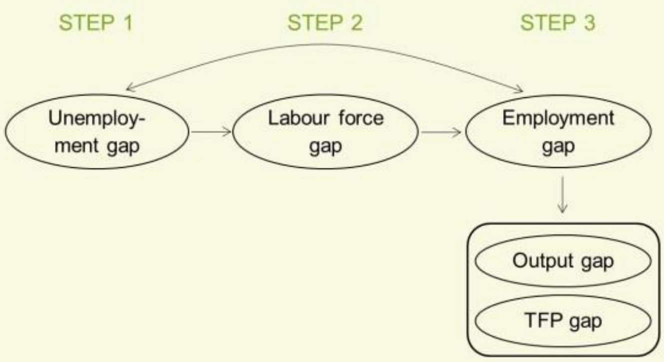
2 MoF methodology

The method used by the MoF to estimate the output gap essentially consists of a system of three separate models that are estimated sequentially in 3 steps, *cf. figure 1*.

- In <u>step 1</u> the unemployment gap is estimated with Philips curve based NAWRU-model.
- The estimated unemployment gap is used as an indicator for cyclical variation in the estimation of the labour force gap in <u>step 2</u>.
- Using the employment gap as an input the output gap and the TFP gap are then estimated simultaneously in <u>step 3</u> of the process, *cf. figure 1*.

The models used in all 3 steps are essentially multi-variate models drawing both on economic relationships (i.e. Philips curve, discouraged worker effect, Okun's law etc.) and the information in key cyclical indicators (real wage growth, capacity utilization etc.). The models are estimated using quarterly data from 1983Q1 to 2013Q4.

Figure 1
Calculation of output and associated gaps in Denmark



Source: Danish Ministry of Finance

Estimation framework and procedure: Step 1 – Unemployment gap

The structural unemployment rate in Denmark is estimated in an economic model based on the expectation-augmented Phillips curve, where the time-varying NAWRU (Non Accelerating Wage Rate of Unemployment) enters as an unobserved state variable.

$$\Delta w_t - \Delta p_t = \alpha_1 + \alpha_2(\Delta w_{t-1} - \Delta p_{t-1}) + \alpha_3(\Delta p^e{}_t - \Delta p_t) + \beta_1 u_t^c + \varepsilon_t \quad (1.1)$$

$$u_t = u_t^* + u_t^c + \varepsilon_t^2 \quad (1.2)$$

$$u_t^* = u_{t-1}^* + \varepsilon_t^3 \quad (1.3)$$

$$u_t^c = \lambda_1 u_{t-1}^c + \lambda_2 u_{t-2}^c + \lambda_{CU} CU_{t-1} + \varepsilon_t^4 \quad (1.4)$$

where Δw is the wage rate, Δp is the inflation rate, Δp^e is the expected inflation rate, u is the actual and u^* is the structural unemployment rate, u_t^c is the unemployment gap, and CU is an indicator for the capacity utilisation in the industrial sector. Expected inflation is specified as the growth rate of HP filtered prices.

The real wage inflationary process in (1.1) is specified as a Phillips curve that includes the information given by the unemployment gap and lagged real wage inflation, given that the inflation process is sticky. Labour market pressure affects nominal wages before they affect prices, and since price inflation in a small open economy is also influenced by exchange rates, import prices and oil prices, the Phillips curve is related to wage inflation rather than price inflation.

Equation (1.2) defines the unemployment rate as being determined as the aggregate of the structural unemployment rate and the unemployment gap. In equation (1.3) the NAWRU is described as an unobserved, stochastic unit root-process.

The unemployment gap in (1.4) follows an AR(2)-process. The indicator for capacity utilization in manufacturing is included as a determinant of the unemployment gap as it provides supplementary information about the cyclical position of the economy, *cf. figure 2*.

Figure 2
Capacity utilization in manufacturing (in deviation from historical mean)

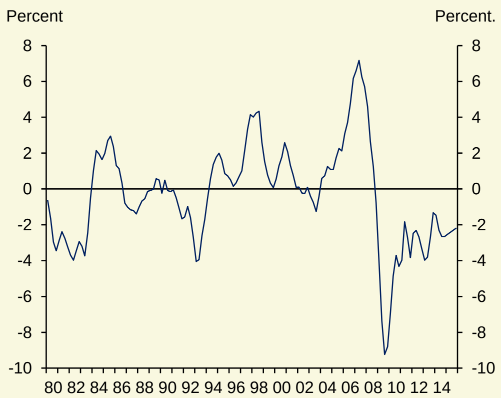
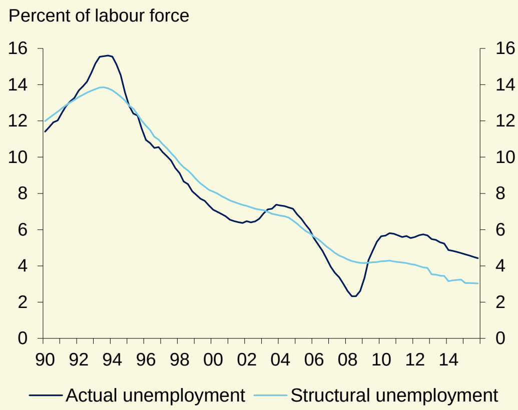


Figure 3
Estimated structural unemployment



Source: Danish Ministry of Finance

The above model framework is designed to give estimates with a high degree of real-time reliability. First of all, the non-revised nature of the indicator for the capacity utilization in the industrial sector reduces the need to revise the model estimates, which improves the real-time reliability of the estimates. The other data input used in the estimation (i.e. the unemployment rate, wage rate, price inflation) are also not subject to any ex post statistical revisions. Secondly, the unemployment statistics in Denmark is based on the Danish administrative unemployment registers, which are highly reliable with a small degree of statistical noise. This also improves real-time reliability of the unemployment gap estimates.

For these reasons the estimated unemployment gap is also used as a key input in determining the cyclical variation in employment and output and is hence used in step 2 and indirectly in step 3 (see below) to help determine cyclical swing and turning points in the labour force gap and the output gap.

Estimation framework and procedure: Step 2 – Labour force gap

The labour force gap is also estimated in a state-space model, where the participation rate is split into a cyclical (e_t^c) and structural component (e_t^*), *cf. figure 5*.

$$e_t = e_t^* + e_t^c \tag{2.1}$$

$$e_t^* = e_t^* + \Delta s_t + \varepsilon_t^1 \tag{2.2}$$

$$e_t^c = \lambda_1 e_{t-1}^c + \lambda_2 u_{t-1}^c + \varepsilon_t^2 \tag{2.3}$$

MINISTRY OF FINANCE

e_t is the participation rate for the 15-64 year age group. Δs_t is an exogenous indicator variable which captures historical structural shifts in the labour force.

The equation (2.3) for the cyclical component models a so-called ”discouraged worker” effect, where the participation rate depends on how favorable job opportunities are. The unemployment gap estimated in step 1 is a used as an indicator for the “discouraged worker” effect in the model. The unemployment gap is assumed to lead the labour force gap.

This has the natural implication that the estimated labour force gap has a close inverse relationship to the unemployment gap. This again implies that the employment gap is closely linked to the unemployment gap.

With the estimates for the structural levels and cyclical gaps in unemployment and the labour force we can calculate the structural employment and an employment gap, *cf. figure 4 and 5*.

Figure 4
Actual and structural employment

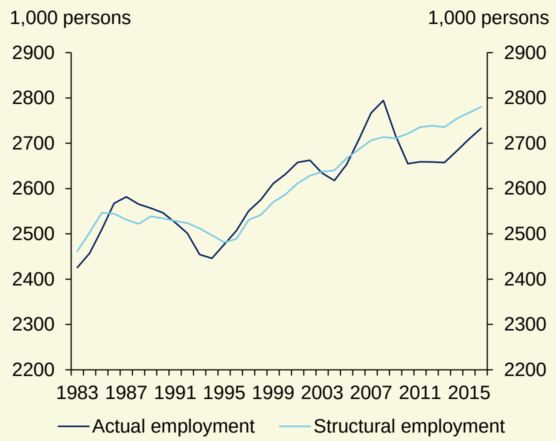
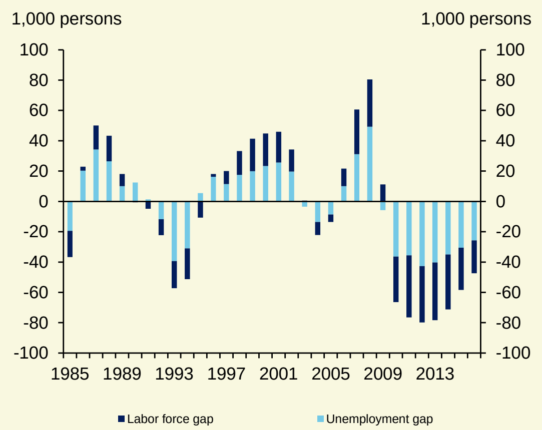


Figure 5
Employment gap – contributions from unemployment and labour force participation



Estimation framework and procedure: Step 3 – Output gap

The output gap model integrates the production function approach and the use of state space models. The starting point is in line with the traditional production function approach the Cobb-Douglas production function with labour (*L*), capital (*K*) and total factor productivity (*TFP*) as factor inputs:

$$Y_t = TFP_t \cdot L_t^{\alpha} K_t^{1-\alpha} \quad (3.1)$$

Labour input is defined as the number of employed and production *(Y)* is measured by total economy gross value added (GVA).

This formulation implies that cycles in the average number of working hours per employed as well as the capacity utilization in capital and labour are a part of *TFP*, which is calculated as the Solow residual. It is assumed that the potential level of capital always equals the actual level, i.e. $K_t = K_t^*$. This expression can then be approximated with the following log-linear relation for the output gap:

$$y_t - y_t^* = (f_t - f_t^*) + \alpha(l_t - l_t^*) \tag{3.2}$$

$$y_t^c = f_t^c + \alpha \cdot l_t^c \tag{3.3}$$

where $y = \log Y$, $f = \log TFP$ and $l = logL$ (as above the superscripts * and *c* denotes the potential/structural level and the cyclical gap respectively).

The observation equations for *y* and *f* split the time series into a structural, cyclical and noise component. By allowing for statistical white noise the model reduces the effect of measurement error and ex-post revisions on the estimated output gap.

$$y_t = y_t^* + y_t^c + \varepsilon_t^y \tag{3.4}$$

$$f_t = f_t^* + f_t^c + \varepsilon_t^f \tag{3.5}$$

Potential GVA (3.6) is modelled by the production function and potential *TFP* (3.7) is modelled as a random walk with a stochastic trend, *cf. figure 6 and 7*.

$$y_t^* = f_t^* + \alpha l_t^* + (1-\alpha)k_t \tag{3.6}$$

$$f_t^* = \gamma_t + f_{t-1}^* + \varepsilon_t^{f^*}, \quad \gamma_t = \gamma_{t-1} + \mu_t^\gamma \tag{3.7}$$

Figure 6
Actual and structural gross value added (GVA)

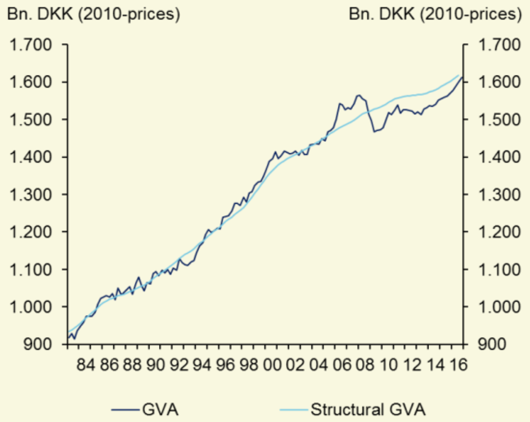
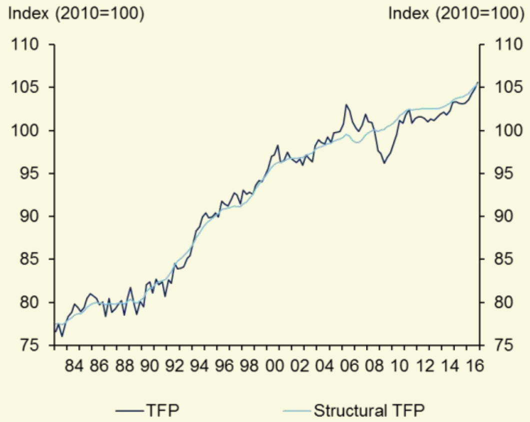


Figure 7
Actual and structural TFP



MINISTRY OF FINANCE

The TFP gap is assumed to follow an AR(2)-process (3.8). Also an observation equation (3.9) is added linking the TFP-gap to the cyclical variation in the capacity utilization in manufacturing (CU). The simultaneous estimation of the output gap and TFP-gap is a key difference compared to traditional methods (see below).

$$f_t^c = \psi_1 f_{t-1}^c + \psi_2 f_{t-2}^c + \varepsilon_t^{f^c} \qquad (3.8)$$

$$CU_t = \theta_1 CU_{t-1} + \theta_2 f_t^c + \varepsilon_t^{CU} \qquad (3.9)$$

On the demand side the employment gap is formulated as an AR(2)-process and related to the output gap as formulated by a Okun's law relationship (3.10). It is assumed that the output gap leads the employment gap. The delay is two quarters, which reflects a historical lag between GDP and employment in Denmark.

$$l_t^c = \lambda_1 l_{t-1}^c + \lambda_2 l_{t-2}^c + \lambda y_{t-2}^c + \varepsilon_t^{l^c} \qquad (3.10)$$

Note that the employment gap is estimated in step 1 and 2. Here it is treated as an exogenous variable which helps identify the cyclical swings in the output gap.

Also, the lag between employment and output gap implicitly implies that the TFP-gap leads the output gap, *cf. figure 8 and 9*. This is in line with common economic intuition, where you would expect productivity to react first during an economic upswing, while hiring of new workers usually lags behind a few quarters. The same mechanism is in effect in a downturn, where labour hoarding behavior implies that firms initially will hesitate to lay off workers, when demand drops.

Figure 8
Output gap, employment gap and TFP gap

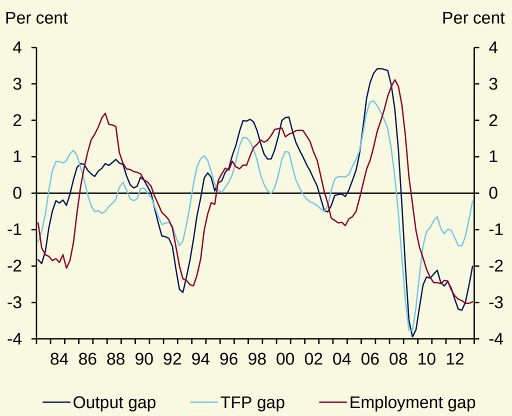
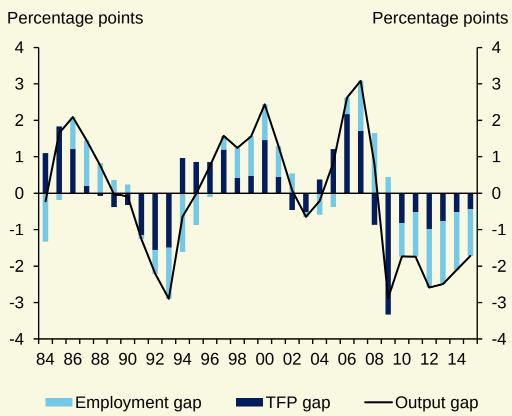


Figure 9
Output gap - Contribution from TFP and employment gap



The inclusion of Okun's law may improve real-time reliability of the output gap estimates. Gross value added (and calculated TFP) is based on national accounts and can be subject to measurement errors and to significant statistical revisions, which can increase estima-

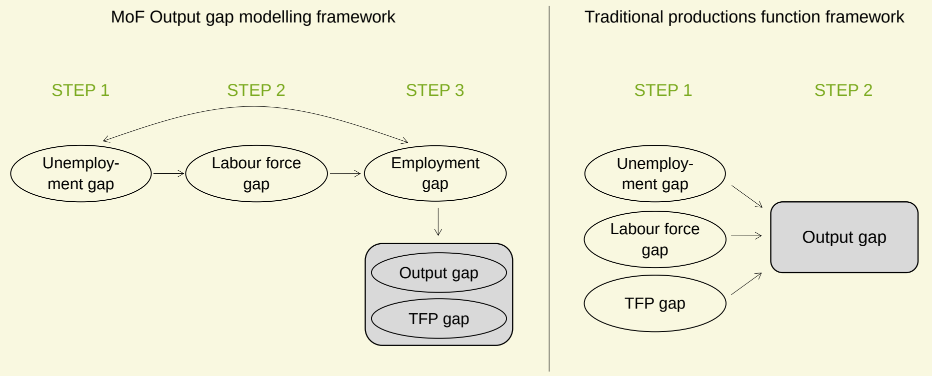
MINISTRY OF FINANCE

tion errors and reduce reliability of output gap estimates. The estimated unemployment gap (and hence also the employment gap) is more immune to these problems as explained above. Okun's law therefore helps "translate" the real-time reliability of unemployment gap to the estimated output gap.

3 Differences between the MoF methodology and traditional production function methods

The 3 steps used in the MoF methodology differs from traditional production function methods, where the calculation of the output gap is often performed in 2 steps, with the gaps in unemployment, labour force and TFP estimated independently, *cf. figure 10*. In the MoF methodology the TFP-gap and the output gap are estimated simultaneously in the same state space model.

Figure 10
Calculation of the structural labour market and output and associated gaps in Denmark



Source: Danish Ministry of Finance

The method used by the MoF also differs from standard methods because the model incorporates Okun's law in the output gap model, thereby establishing an explicit link between the employment gap and the output gap in line with stylized facts.

Moreover, the method relies entirely on state-space estimation techniques using the Kalman filter, rather than the more commonly used Hodrick-Prescott-filter (HP)[1]. Compared to the HP-filter the Kalman-filter offers several empirical advantages:

- The economic relationships of the variables under consideration are taken into account, such that the trend and cycle of the production inputs have a clear economic interpretation.

[1] In the methodology used by the EU-commission the HP-filter is used to estimate the trend participation rate and the labor force gap.

- End-point problems tend to be smaller when using the Kalman Filter.
- The division between trend and cycle is estimated based on the data rather than using ad-hoc assumptions to determine the degree of smoothness in the trend.[2]
- By including error terms in observation equations the state space model can in principle control for data noise.
- The method makes it possible to estimate confidence bands for the estimated coefficients.

In the following it will be argued that the above model features improve both the precision and the real-time reliability of the output gap estimates compared to traditional methods.

4 Estimates of output gaps for Denmark and revisions over time

The method used by the Danish Ministry of Finance displays significant real-time reliability. This is indicated by the stability of the estimated output gap from 2003 and 2011 at various points in time, cf. table 1. The table shows the estimates of years covered by historical data and therefore the revision are influenced by forecast errors.

Table 1
Estimates of the output gap for Denmark in 2003, 2005, 2007, 2009 and 2011 across time

			MoF		
Spring forecast	**2003**	**2005**	**2007**	**2009**	**2011**
2004	-1.1				
2005	-0.8				
2006	-0.3	0.9			
2007	-0.5	0.7			
2008	-0.8	0.2	3.1		
2009	-0.5	0.7	2.9		
2010	-0.6	1.1	3.2	-2.9	
2011	-0.8	1.0	2.8	-3.2	
2012	-0.8	0.8	3.0	-2.9	-1.9
2013	-0.7	0.8	3.0	-3.0	-1.9
2014	-0,7	0,8	3,1	-2,8	-1,7
Total revision, pct.points	**0.4**	**-0.1**	**0.0**	**0.1**	**0.2**

[2] Due to the so-called "pile-up problem" (see Stock and Watson (1998)) state-space models estimated with the Kalman Filter in many cases have difficulty with the estimation of the signal-to-noise ratio (i.e. the ratio between the standard deviations of error terms in the state equation and the observation equation). In other words it is difficult to separate variance in the data from the dynamics of the unobserved states by standard maximum likelihood techniques. Therefore it is often necessary to set the signal-to-noise ratio a priori.

MINISTRY OF FINANCE

Note: The table indicates the estimates for the 2003, 2005, 2007, 2009 and 2011 in the spring forecast each year.
Source: Danish Ministry of Finance (MoF)

Table 2 compares to similarly timed output gap estimates from the OECD and the EC (DG ECFIN). As it is apparent from the table, the output gap estimates from the MoF displays significant stability compared to the estimates of the other institutions, especially the estimates of how much output was above the structural level in 2007 before the financial crisis.

Even though the evidence in table 2 is quite striking one should be careful in drawing too strong conclusions from this comparison.

First of all, the estimates are sensitive to changes in the methodology. The MoF models have been reestimated every year, but the methodology has overall remained the same over the years with only a few changes. It is not clear whether the same model stability applies to the estimates of the the EC and OECD. Secondly, the Ministry of Finance naturally have more ressources dedicated to analyzing the Danish economy which should imply more well-specified models customized for the Danish economy.

Table 2
Estimates of the output gap for Denmark in 2003, 2005, 2007, 2009 and 2011 across time

	Springs forecast					Total revision, percentage points
2003	2004	2005	2006	2007	2008	
MoF	-1.1	-0.8	-0.3	-0.5	-0.8	0.3
ECFIN	-0.7	-1.0	-1.6	-1.6	-1.5	-0.8
OECD	-1.3	-1.7	-2.1	-2.0	-1.9	-0.6
2005	2006	2007	2008	2009	2010	
MoF	0.9	0.7	0.2	0.7	1.1	0.2
ECFIN	-0.8	-0.5	-0.4	0.8	0.8	1.6
OECD	-0.7	-0.3	-0.1	0.8	0.2	0.9
2007	2008	2009	2010	2011	2012	
MoF	3.1	2.9	3.2	2.8	3.0	0.2
ECFIN	0.7	2.4	2.6	3.1	3.3	2.6
OECD	2.1	2.4	2.1	2.2	4.6	2.5
2009	2010	2011	2012	2013		
MoF	-2.9	-3.2	-2.9	-3.0		-0.1
ECFIN	-5.1	-5.6	-5.0	-4.7		0.4
OECD	-6.5	-6.6	-4.2	-3.6		2.9
2011	2012	2013	2014			
MoF	-1.9	-1.9	-1.7			0.2
ECFIN	-3.1	-3.5	-3.6			-0.5
OECD	-3.2	-2.2	-2.2			1.0

Note.: The table indicates the estimates for the 2003, 2005, 2007, 2009 and 2011 in the spring forecast each year for the following 3-5 years.

Source: OECD, European Commission (ECFIN) and Danish Ministry of Finance (MoF)

5 Analyzing reliability using synthetic data

The evidence of past revisions of output gap estimates also does not give much indication as to what model-properties that may be the reason for what seems to be higher reliability of the MoF-estimates.

In an attempt to identify which model characteristics that are important for reliability, the remainder of this paper applies the methodologies of MoF and the EC on synthetic data and compares the estimates. The approach allows us to test the different model features in a controlled environment and to test how sensitive output gap estimates are to data noise and model misspecification.

In the following we will focus on reliability in a broad sense analyzing both precision and real-time reliability. Precision is defined how close the estimated values are to the "true" values. Real-time reliability is defined as the stability of the estimates.

To limit the scope of this paper, we focus on the estimation of TFP gaps and output gaps. Therefore the analysis will focus on the differences in methodology regarding these estimates. These differences are summed up in table 3.

Table 3
Key differences between MoF methodology and traditional production function method

Ministry of Finance	Traditional production function method
Simultaneous estimation of output gap and TFP-gap.	Separate and independent estimation of unemployment gap, labour force gap and TFP-gab. These are then com-bined to construct an output gap.
In addition to the production function the model framework includes a lagged relationship between output gap and employment gap (via Okun's law).	Output gap and employment gap are only related via the production function.
Allows for error residuals in observation equations	Data noise will potentially affect the estimates for both the output gap and potential output

We will not analyze how different modelling of structural unemployment and structural labour force affects precision and real-time reliability. This also reflects that there is less consensus about how to model the labour market across countries, for example which formulation of the Philips curve is most appropriate for different types of labour markets. Hence, it is difficult for international institutions to find a uniform modelling framework for the labour market that can be applied to all countries.

The production function approach used to estimate TFP and output gaps – whether HP-filter, single or multi-variate based methods –is on the contrary more broadly accepted.

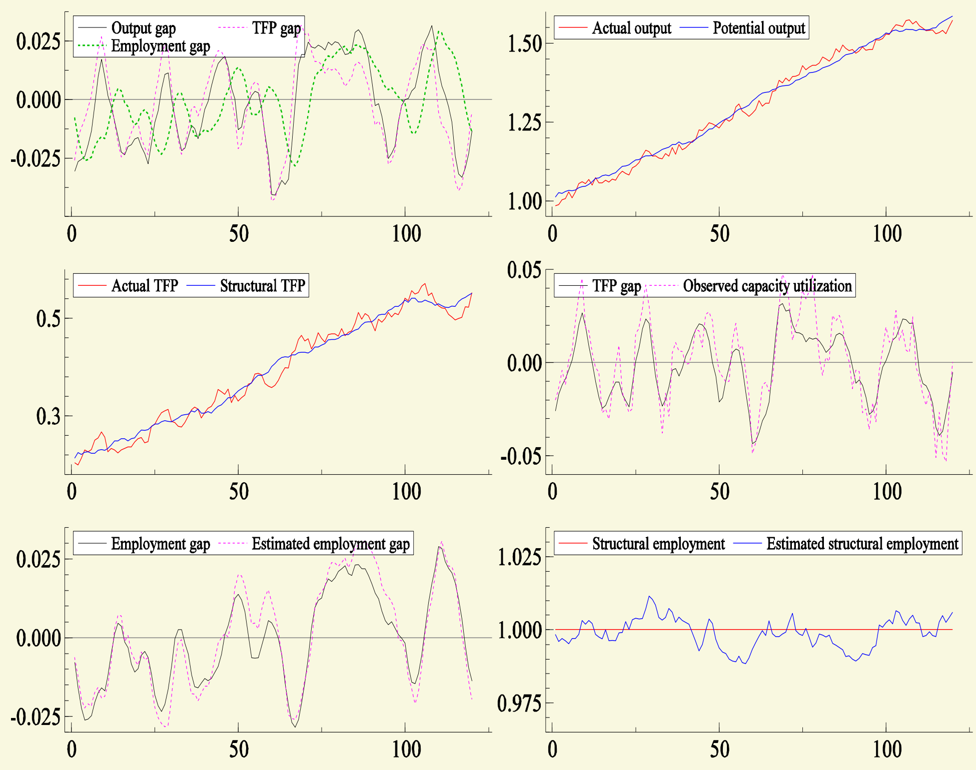
It is therefore assumed that both models use the same estimates for structural employment and the employment gap and differences in precision and reliability are thus alone the result of errors in the estimated TFP and output gaps.

Synthetic data based on the Danish economy

The synthetic data used in the analysis is constructed from small time series model of the Danish economy. Time series for actual and structural levels as well as cyclical gaps for output, TFP and employment are constructed for 100 quarters (25 years), *cf. figure 4*. See appendix for details on time series equations and parameter values.

Figure 4

Example of synthetic data for actual and structural levels for output, TFP, employment etc.



Source: Danish Ministry of Finance

MINISTRY OF FINANCE

The two models

The analysis compares two models. The MoF-model, where the output gap and the TFP gap are estimated simultaneously and Okun's Law is included as an extra observation equation, *cf. table 4*. And a model based on the traditional production function method, where the TFP-gap is estimated separately and then together with the employment gap used to construct an estimate for the output gap.

As mentioned it is assumed that both models use the same estimates for structural employment and the employment gap.

Table 4
Reduced form estimation models for analysis with synthetic data

Ministry of Finance	Traditional production function method
$\hat{l}_t^*$ and $\hat{l}_t^c$ treated as exogenous variables	**$\hat{l}_t^*$ and $\hat{l}_t^c$ treated as exogenous variables**
$\widehat{y_t^*}$, $\widehat{y_t^c}$, $\widehat{f_t^*}$ and $\widehat{f_t^c}$ are estimated simultaneously with the following model:	**$\widehat{f_t^*}$ and $\widehat{f_t^c}$ are estimated with the following model:**
State equations:	*State equations:*
$f_t^c = \psi_1 f_{t-1}^c + \psi_2 f_{t-2}^c + \varepsilon_t^{f^c}$	$f_t^c = \psi_1 f_{t-1}^c + \psi_2 f_{t-2}^c + \varepsilon_t^{f^c}$
$f_t^* = \gamma + f_{t-1}^* + \varepsilon_t^{f^*}$	$f_t^* = \gamma + f_{t-1}^* + \varepsilon_t^{f^*}$
$y_t^c = f_t^c + \alpha \hat{l}_t^c$	*Observation equations:*
$y_t^* = f_t^* + \alpha \hat{l}_{t-1}^* + (1-\alpha) k_t$	$CU_t = \theta f_t^c + \varepsilon_t^{CU}$
Observation equations:	$f_t = f_t^* + f_t^c$
$CU_t = \theta f_t^c + \varepsilon_t^{CU}$	
$f_t = f_t^* + f_t^c + \varepsilon_t^f$	**$\widehat{y_t^*}$ and $\widehat{y_t^c}$ are calculated separately with production function identities:**
$y_t = y_t^* + y_t^c + \varepsilon_t^y$	$\hat{y}_t^c = \hat{f}_t^c + \alpha \hat{l}_t^c$
$\hat{l}_t^c = \lambda_1 \hat{l}_{t-1}^c + \lambda_2 \hat{l}_{t-2}^c + \lambda_3 y_{t-2}^c + \varepsilon_t^{l^c}$	$\hat{y}_t^* = \hat{f}_t^* + \alpha \hat{l}_{t-1}^* + (1-\alpha) k_t$

Precision of the MoF compared to traditional production function methods

Using synthetic data it is possible to test how close the estimates are to the "true" values of the output gap. Figure 8 and 9 depicts the estimated and "true" values of potential output and the output gap.

The precision of the two models are evaluated with the Root-Mean-Squared Errors (RMSE). The RMSE show that the MoF methodology is more precise in estimating the output gap – the estimates are on average about 0.1 percentage point closer to the "true" value, *cf. table 4*.

That the MoF model is more accurate than the traditional production function approach (PF-model) is not surprising. By construction the inclusion of Okun's law as part of the time series model creating the synthetic data, implies an advantage for the MoF-model, because it exploits the information in Okun's law. Therefore it will be more relevant to use the RMSEs to identify how sensitive the models are to alterations of the model, for instance in terms of changing the standard deviation of the error-terms in the data.

Figure 8
MoF – estimated output gap on synthetic data

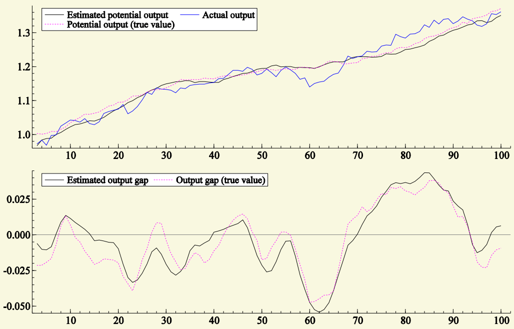
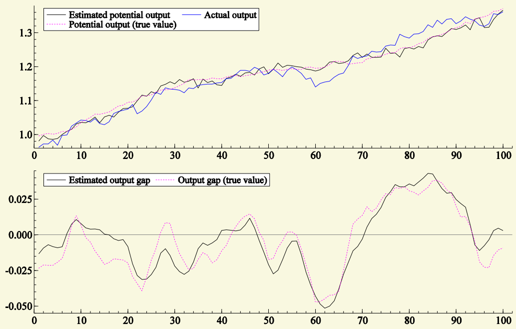


Figure 9
Traditional production function approach – estimated output gap on synthetic data



Below RMSE are calculated for a number of alternative data specifications. In all cases, the MoF-model displays a somewhat smaller RMSE than is the case for the PF-model.

The models are not equally sensitive to all modifications of the variance terms. Both models are relatively insensitive to changes in the variance of the ε_t^f term (appendix equation 7.8), as can be seen from table 5 below. The PF-model has by definition no reaction to changes in the noise term σ^y (appendix equation 7.7). The MoF model reacts, but with little sensitivity, to changes in σ^y.

Table 5
Sensitivity to statistical noise in the data (y and f)

RMSE	PF-model	MoF-model
Baseline scenario, $\sigma^f = 0.0070$	**0.0082**	**0.0071**
Increased noise in f_t, $\sigma^f = 0.0140$	0.0087	0.0073
Decreased noise in f_t, $\sigma^f = 0.0035$	0.0077	0.0069
Baseline scenario, $\sigma^y = 0.070$	**0.0082**	**0.0071**
Increased noise in y_t, $\sigma^y = 0.0140$	0.0082	0.0072
Decreased noise in y_t, $\sigma^y = 0.0035$	0.0082	0.0070

Note: Changes in the ε_t^y term will by definition not affect the EC-Model.
Source: Danish Ministry of Finance

A much stronger reaction for both models is achieved when manipulating the standard deviation of the error term in the capacity utilization (CU) relationship (Equation 7.5 in the appendix). Both models react to changes in σ^{CU} with greater magnitude in the change

MINISTRY OF FINANCE

in RMSE than for error-terms above. The PF-model reacts both absolute and relatively stronger to the change as can be seen from table 6. The result shows that the precision of the PF-model is highly dependent of how close the indicator for capacity utilization is linked to the "true" value of the TFP-gap.

The MoF-model also used the information in Okun's law and is therefore more immune to noise in the capacity utilization relationship.

Table 6
Sensitivity to noise in the CU

RMSE	EC-model	MoF-model
Baseline scenario, $\sigma^{CU} = 0.0100$	**0.0082**	**0.0071**
Increased noise in CU_t, $\sigma^{CU} = 0.0200$	0.0126	0.0089
Decreased noise in CU_t, $\sigma^{CU} = 0.0050$	0.0055	0.0054

Source: Danish Ministry of Finance

The precision of both models are naturally very sensitive to how accurate the employment gap is estimated (measured by the value of $\sigma^{\hat{l}^c}$). But even though the MoF model relies more heavily on the estimates for l^c by using Okun's Law, the MoF-model reacts only slightly stronger to changes in the noise of the employment variable ($\hat{l}^c$ is the variable used in place of the true l^c generated in the data).

Table 7
Sensitivity to how accurate the employment gap is estimated

RMSE	PF-model	MoF-model
Baseline scenario, $\sigma^{\hat{l}^c} = 0.0025$	**0.0082**	**0.0071**
Increased noise in $\hat{l}_t^c$, $\sigma^{\hat{l}^c} = 0.0050$	0.0100	0.0092
Decreased noise in $\hat{l}_t^c$, $\sigma^{\hat{l}^c} = 0.00125$	0.0077	0.0066

Source: Danish Ministry of Finance

Since Okun's Law is a key difference between the models, it is highly relevant to evaluate how sensitive the MoF-model is to changes in the relationship. These changes do naturally not affect the precision of the PF-model.

In table 8 presents the RMSE for a number of changes to the Okun's Law equation.

MINISTRY OF FINANCE

Firstly, we change the standard deviation of the error term in Okun's law. This only has a marginal effect on precision. Then we modify how long the length of the lag of output on employment is in the synthetic data. As is clear from the table changes have little effect on the resulting RMSE of the model. Similar conclusion comes when imposing structural breaks in the model halfway through the timespan; very little happens to the model's RMSE. Changing the process that generates the error-term in the Okun's Law relationship, $\varepsilon_t^{l^c}$, to be an MA-process also does little to disturb the fit of the model.

This is a somewhat surprising result, as one would expect misspecification of Okun's law to have a large impact on precision. However, because the CU-relationship is included in the model a poorer fit of Okun's law will lead the model to put more emphasis on the CU-relationship when estimating the TFP-gap and the output gap. Also the AR(2) formulation of Okun's law seems general enough to encompass some degree of misspecification in lag length etc.

Table 8

Sensitivity to changes in the noise in the Okun's Law and specification of the relationship

	MoF-model
	$\sqrt{MSE}$
Baseline scenario, $\sigma^{l^c} = 0.0019$	**0.0071**
Increased noise in l_t^c, $\sigma^{l^c} = 0.0038$	0.0073
Decreased noise in l_t^c, $\sigma^{l^c} = 0.00085$	0.0069
Lag length = 2 in the term $\lambda_3 y_{t-2}^c$, baseline scenario	**0.0071**
Lag length = 1	0.0071
Lag length = 4	0.0073
Lag length = 8	0.0076
Sensitivity to breaks in parameters occurring at t=50	**0.0071**
Parameter λ_1 - 0.2 and λ_2 - 0.2	0,0074
Parameter λ_3 -0.05	0,0075
Parameter λ_3 -0.10	0,0077
Change noise term to be MA: $\varepsilon_t^{l^c} = \rho * \varepsilon_{t-1}^{l^c} + \varepsilon_t^{MA}$, baseline $\rho = 0$	**0,0071**
$\rho = 0.4$	0,0072
$\rho = 0.9$	0,0074

Source: Danish Ministry of Finance

Real-time reliability of the MoF compared to traditional production function methods

Next, we compare the real-time reliability of the MoF and EC models on the artificial data. This is done by generating 1000 samples each of which is 120 periods long. First, both models are run on the first 100 periods of each sample. For each of the models and samples the deviation from the true value of the TFP-gap at the 100th period (in this case

MINISTRY OF FINANCE

the end point) is stored.[3] This is called the real-time deviation. Second, the models are run on the full 120 periods. Again, the deviation from the true value of the TFP-gap at the 100th period (this time an interior point of the sample) is stored for each of the models and samples. This is called the updated deviation in the following.

Table 4
Root-Mean-Squared Error of TFP gap estimate at period 100

	EC-model	MoF-model
RMSE (without real-time noise)		
Real-time deviation	0,0125	0,0123
Updated deviation	0,0123	0,0121
RMSE (with real-time noise)		
Real-time deviation	0,0134	0,0128
Updated deviation	0,0123	0,0121

Note: The RMSE is calculated across the samples (out of 1000) where both models converged, which they did for 757 samples in the case without real-time noise and 758 samples (out of 1000) in the case with real-time noise. I.e. the RMSE is not calculated across different samples for the two models in order to avoid selection issues.
Source: Danish Ministry of Finance

This exercise is conducted both with and without adding extra noise to the observed variables. When adding extra noise, an IID error term is added to the four last periods of the sample. That is, when running the models on the first 100 periods, extra noise is added to periods 97-100, and when running the models on the full 120 periods, extra noise is added to periods 117-120.[4] This is referred to as real-time noise, and is included in order to simulate, that the observed values for the last periods of the sample are usually preliminary statistics and thus subject to subsequent revisions. With real-time noise included, the observed variables for period 97-100 will thus contain less noise when we estimate on the full 120 periods than when we estimate on the first 100 periods.

As seen from table 4, the MoF-model has lower RMSE both for the real-time deviation and the updated deviation. This is true both with and without real-time noise in the last sample periods. Thus the MoF-model performs better in real-time estimation of the output gap, although the difference is small.

[3] Difference in the estimate of the TFP-gap is the only source of difference in the estimate of the output gap – at least when calculating the output gap based on the estimated values of the TFP-gap and the observed value of the employment gap.
[4] In particular, in the last period noise is added to f and $CU$with the same variance as ε_t^f and ε_t^{CU} respectively. In the second, third and fourth to last periods, noise is added with variance of ½, ¼th and 1/8th the variance of ε_t^f and ε_t^{CU} respectively.

6 Conclusion

The historical evidence on past revisions of output gap estimates show that the methodology used by the Danish Ministry of Finance displays higher stability than estimates based on the traditional production function approach.

The analysis with synthetic data indicates that the multi-variate approach with the inclusion of Okun's law and simultaneous estimation of the TFP and output gaps are key model features in explaining why the precision and real-time reliability is higher for the MoF output gap estimates. Also the inclusion of error terms in the MoF-model to control for statistical noise in both output and TFP will tend to improve estimates.

The multi-variate approach can in principle exploit more information in cyclical indicators and in reduced form economic relationships. In many cases it will therefore be superior to single-variate filtering methods that often used in the literature.

A drawback to the multi-variate approach is that is adds considerable complexity to the Kalman Filter estimation problem and be difficult to find sensible parameter estimates.

7 Appendix – Construction of artificial data

The data is based on a time series model for a simplified economy, see equations 7.1-7.8. Time series for actual and structural levels as well as cyclical gaps for output, TFP and employment are constructed for 100 quarters (25 years). The error terms are randomly drawn from a normal distribution. The parameter and standard deviations used are estimated on Danish data, *cf. table A1*.

$$f_t^c = \psi_1 f_{t-1}^c + \psi_2 f_{t-2}^c + \varepsilon_t^{f^c} \quad (7.1)$$

$$l_t^c = \lambda_1 l_{t-1}^c + \lambda_2 l_{t-2}^c + \lambda_3 y_{t-2}^c + \varepsilon_t^{l^c} \quad (7.2)$$

$$\hat{l}_t^c = l_t^c + \varepsilon_t^{l^c} + \rho\varepsilon_{t-1}^{l^c} \quad (7.2)$$

$$f_t^* = \gamma + f_{t-1}^* + \varepsilon_t^{f^*} \quad (7.3)$$

$$y_t^* = f_t^* + \alpha l_{t-1}^* + (1-\alpha)k_t \quad (7.4)$$

$$CU_t = \theta f_t^c + \varepsilon_t^{CU} \quad (7.5)$$

$$y_t^c = f_t^c + \alpha l_t^c \quad (7.6)$$

$$y_t = y_t^* + y_t^c + \varepsilon_t^y \quad (7.7)$$

$$f_t = f_t^* + f_t^c + \varepsilon_t^f \quad (7.8)$$

Table A1
Parameter estimates

	Estimate
Model parameters	
λ_1	1,4673
λ_2	-0,5974
λ_y	0,1127
θ_1	0,5639
θ_2	1,1592
ψ_1	1,6098
ψ_2	-0,7082
ρ	0,9000
Standard deviation of error residuals	
σ^{f^c}	0,0043
σ^{f^*}	0,0044
$\sigma^{\hat{l}^c}$	0,0025
σ^f	0,0071
σ^y	0,0073
σ^{q^c}	0,0019
σ^{CU}	0,0050

[Noget generelt skriv om FMs arbejdspapirer]

www,fm,dk